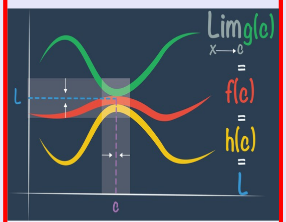
Math 261

Spring 2022

Lecture 24



Class QZ 13

1) Evaluate $\int_0^2 (8x^3+3x^2)\,dx$

$$=\left(\frac{8x^4}{4}+\frac{3x^3}{3}\right)\Big|_0^2=(2x^4+x^3)\Big|_0^2=2\cdot2^4+2^3=\boxed{40}\ \checkmark$$

2) Find $\int x \sec x^2 \tan x^2\,dx$

$u=x^2$

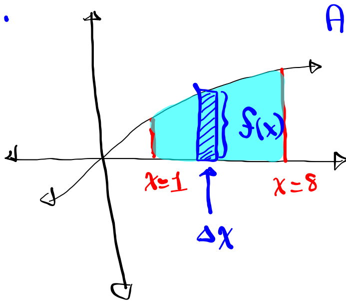
$du=2x\,dx$

$\frac{du}{2}=x\,dx$

$$=\int \sec u \tan u \cdot \frac{du}{2}=\frac{1}{2}\sec u+C$$

$$=\boxed{\frac{1}{2}\sec x^2+C}\ \checkmark$$

Find the area in QI, below $f(x)=\sqrt[3]{x}$ for $1\le x\le 8$.



$$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\Delta x=\frac{b-a}{n}=\frac{8-1}{n}=\frac{7}{n}$$

$$x_i=a+i\Delta x$$

$$x_i=1+i\cdot\frac{7}{n}$$

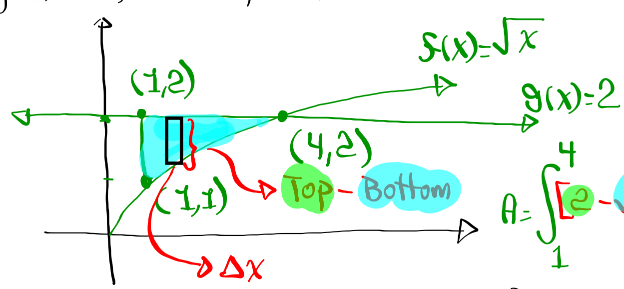
$$x_i=\frac{7i}{n}+1$$

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{7i}{n}+1}\cdot\frac{7}{n}$$

$$=\int_{a=1}^{b=8} f(x)\,dx=\int_1^8\sqrt[3]{x}\,dx=\int_1^8 x^{\frac{1}{3}}\,dx=\left.\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_1^8$$

$$=\left.\frac{x^{4/3}}{4/3}\right|_1^8=\frac{3}{4}\left[8^{4/3}-1^{4/3}\right]=\frac{3}{4}\left[2^4-1^4\right]=\frac{3\cdot 15}{4}=\boxed{\frac{45}{4}}$$

Find the area enclosed by $f(x)=\sqrt{x}$, $g(x)=2$, $x=1$, and $x=4$.



$$A=\int_1^4\left[2-\sqrt{x}\right]dx \qquad x^{\frac{1}{2}}$$

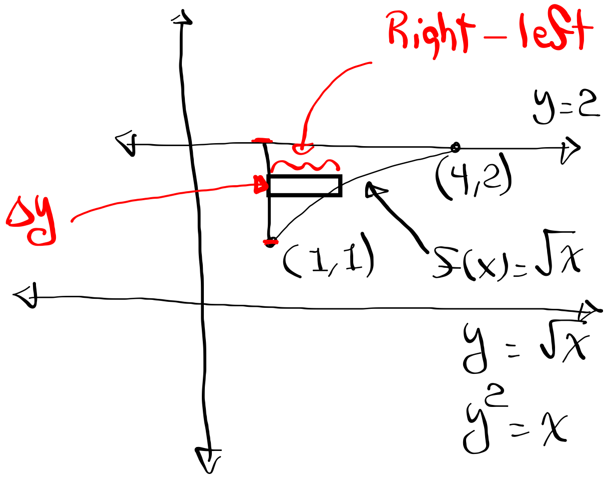
$$=\left.\left(2x-\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_1^4=\left.\left(2x-\frac{x^{3/2}}{3/2}\right)\right|_1^4=\left.\left[2x-\frac{2}{3}x\sqrt{x}\right]\right|_1^4$$

$$=\left(2(4)-\frac{2}{3}\cdot 4\sqrt{4}\right)-\left(2\cdot 1-\frac{2}{3}\cdot 1\sqrt{1}\right)$$

$$=8-\frac{16}{3}-2+\frac{2}{3}=6-\frac{14}{3}=\frac{18}{3}-\frac{14}{3}=\boxed{\frac{4}{3}}\checkmark$$

$x = y^2 \qquad x = 1$

Right – left



$y = \sqrt{x}$

$y^2 = x$

$$A = \int_1^2 [y^2 - 1]\, dy$$

$$= \left(\frac{y^3}{3} - y\right)\Big|_1^2$$

$$= \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right)$$

$$= \frac{8}{3} - 2 - \frac{1}{3} + 1$$

$$= \frac{7}{3} - 1 = \frac{7}{3} - \frac{3}{3} = \boxed{\frac{4}{3}} \checkmark$$

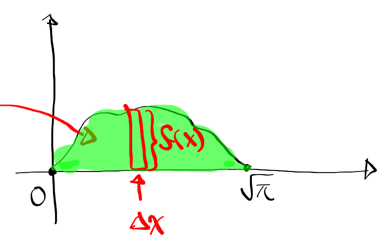
Graph $f(x) = x \sin x^2$ for $0 \le x \le \sqrt{\pi}$

$x \ge 0 \quad \sin x^2 \ge 0 \quad f(x) \ge 0$

$\sqrt{\pi} < \pi$

$0 \le x \le \sqrt{\pi} \Rightarrow QI$

Find this area.



$$A = \int_0^{\sqrt{\pi}} x \sin x^2\, dx$$

Let $u = x^2$ $\qquad \frac{du}{dx} = 2x$

$x = 0 \quad u = 0 \qquad du = 2x\,dx$

$x = \sqrt{\pi} \quad u = \pi \qquad \frac{du}{2} = x\,dx$

$$= \int_0^{\pi} \sin u \cdot \frac{du}{2} = \frac{1}{2}\left[-\cos u \Big|_0^{\pi}\right]$$

$$= \frac{-1}{2}[\cos \pi - \cos 0]$$

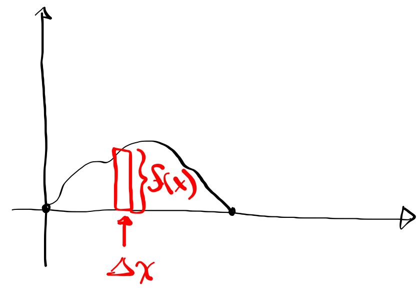
$$= \frac{-1}{2}[-1 - 1] = \frac{-1}{2} \cdot (-2) = \boxed{1}$$

Find the area below $f(x) = x\sqrt{1-x^2}$, above $y=0$ for $0 \le x \le 1$.

$f(0)=0$

$f(1)=0$

$f(x) \ge 0$ for $0 \le x \le 1$



$$A = \int_0^1 x\sqrt{1-x^2}\,dx$$

$u = 1-x^2 \qquad \frac{du}{dx} = -2x$

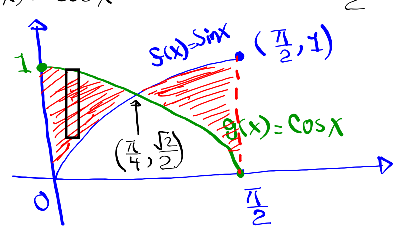
$x=0 \to u=1$

$x=1 \to u=0 \qquad \frac{du}{-2} = x\,dx$

$$A = \int_1^0 \sqrt{u}\,\frac{du}{-2} = \frac{-1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^0 = \frac{-1}{3}\left[0^{3/2} - 1^{3/2}\right]$$

$$= \frac{-1}{3}(-1) = \boxed{\frac{1}{3}}$$

Find the area bounded by $f(x) = \sin x$, $g(x) = \cos x$ for $0 \le x \le \frac{\pi}{2}$.



$$A = \int_0^{\pi/4} [\cos x - \sin x]\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\underbrace{\sin 0}_{0} + \underbrace{\cos 0}_{1}\right) - \left[\underbrace{\cos\tfrac{\pi}{2}}_{0} + \underbrace{\sin\tfrac{\pi}{2}}_{1}\right] + \left[\cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}\right]$$

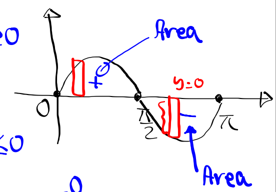
$$= \underbrace{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)}_{\sqrt{2}} - 1 - 1 \underbrace{\left(+\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)}_{\sqrt{2}}$$

$$= \boxed{2\sqrt{2} - 2}$$

Find the area bounded by $f(x) = \sin x \cos x$ for $0 \le x \le \pi$ with x-axis.

$0 \le x \le \frac{\pi}{2}$ $\quad$ $\sin x \ge 0$, $\cos x \ge 0$ $\Rightarrow f(x) \ge 0$

$\frac{\pi}{2} \le x \le \pi$ $\quad$ $\sin x \ge 0$, $\cos x \le 0$ $\Rightarrow f(x) \le 0$



$f(0) = 0$

$f(\pi/2) = 0$

$f(\pi) = 0$

IS $\int_0^{\pi} \sin x \cos x \, dx = \int_0^0 u \, du = 0$

$u = \sin x$ $\quad$ $du = \cos x \, dx$

$x = 0$ $\quad$ $u = 0$

$x = \pi$ $\quad$ $u = 0$

$\int x^n dx = \boxed{\frac{x^{n+1}}{n+1} + C}$

$A = \int_0^{\frac{\pi}{2}} \sin x \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} - \sin x \cos x \, dx =$

$= 2 \int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = 2 \int_0^1 u \, du = 2 \cdot \frac{u^2}{2} \Big|_0^1 = \boxed{1}$

$u = \sin x$ $\quad$ $du = \cos x \, dx$ $\quad$ $x = 0 \Rightarrow u = 0$, $x = \pi/2 \Rightarrow u = 1$

Evaluate

$\int \cos(\pi x) \, dx$

$u = \pi x$

$\frac{du}{dx} = \pi$

$\frac{du}{\pi} = dx$

$= \int \cos u \, \frac{du}{\pi}$

$= \frac{1}{\pi} \sin u + C = \boxed{\frac{1}{\pi} \sin(\pi x) + C}$

$\int \frac{1}{(1-5x)^6} dx$ $\quad$ $u = 1 - 5x$, $du = -5 dx$ $\Rightarrow \frac{du}{-5} = dx$

$= \int \frac{1}{u^6} \cdot \frac{du}{-5} = \frac{-1}{5} \int u^{-6} du = \frac{-1}{5} \cdot \frac{u^{-6+1}}{-6+1} + C$

$= \frac{-1}{5} \cdot \frac{u^{-5}}{-5} + C$

$= \frac{1}{25} \cdot \frac{1}{u^5} + C$

$= \boxed{\frac{1}{25(1-5x)^5} + C}$

$$\int \frac{\sec^2\left(\frac{1}{x}\right)}{x^2}\,dx \qquad u = \frac{1}{x}$$

$$\frac{du}{dx} = \frac{-1}{x^2} \qquad \frac{du}{-1} = \frac{1}{x^2}dx$$

$$= \int \sec^2 u \,\frac{du}{-1} = -\int \sec^2 u\,du = -\tan u + C$$

$$= \boxed{-\tan\left(\tfrac{1}{x}\right) + C}$$

Evaluate $\displaystyle\int \frac{2x}{1+x^2}\,dx$

If $u = x^2$

$du = 2x\,dx$

$$I = \int \frac{1}{1+u}\,du$$

If $u = 1 + x^2$

$du = 2x\,dx$

$$I = \int \frac{1}{u}\,du =$$

Power rule works for $n \neq -1$

$\int u^{-1}\,du$

wait for Calc II.

$$\int e^u\,du = e^u + C,$$

Calc II

Find $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$$I = \int e^u\,du = e^u + C$$

$$= \boxed{e^{\tan x} + C}$$

find $\int (x+1)\sqrt{x^2+2x}\,dx = \int \sqrt{u}\,\frac{du}{2}$

$u = x^2+2x$

$du = (2x+2)dx$

$du = 2(x+1)dx$

$\frac{du}{2} = (x+1)dx$

$= \frac{u^{3/2}}{\frac{3}{2}}\cdot\frac{1}{2} + C$

$= \frac{1}{3}u\sqrt{u} + C$

$= \boxed{\frac{1}{3}(x^2+2x)\sqrt{x^2+2x} + C}$

$\int x^2\sqrt{1+x}\,dx$

Let $u = 1+x \rightarrow x = u-1$

$du = dx$

$I = \int (u-1)^2\cdot\sqrt{u}\,du = \int (u^2-2u+1)\sqrt{u}\,du$

$= \int (u^2\sqrt{u} - 2u\sqrt{u} + \sqrt{u})du$

$= \int [u^{5/2} - 2u^{3/2} + u^{1/2}]du$

$\vdots$

IS $u = \sqrt{1+x}$ $\rightarrow 2u\,du = dx$

$u^2 = 1+x$ $\rightarrow x = u^2-1$

$\int x^2\sqrt{1+x}\,dx = \int (u^2-1)^2\cdot u\cdot 2u\,du$

$= \int (u^4-2u^2+1)\cdot 2u^2\,du$

$= \int (2u^6 - 4u^4 + 2u^2)\,du$

$\vdots$

Find $\int x^3\sqrt{1+x^2}\,dx$

Let $u = 1+x^2$ $\quad du = 2x\,dx$ $\quad \frac{du}{2} = x\,dx$

$x^3 = x^2 \cdot x$ $\quad x^2 = u-1$

$$\int x^3\sqrt{1+x^2}\,dx = \int x^2 \cdot x\sqrt{1+x^2}\,dx$$

$$= \int (u-1)\cdot\sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\int (u^{3/2} - u^{1/2})\,du$$

Evaluate $\int_0^{\pi} \sec^2\left(\frac{x}{4}\right)dx$

Let $u = \frac{x}{4}$ $\quad 4u = x$ $\quad 4\,du = dx$

$x = 0 \rightarrow u = 0$

$x = \pi \rightarrow u = \frac{\pi}{4}$

$$\int_0^{\pi} \sec^2\frac{x}{4}\,dx = \int_0^{\pi/4} \sec^2 u \cdot 4\,du = 4\int_0^{\pi/4}\sec^2 u\,du$$

$$= 4\cdot \tan u\Big|_0^{\pi/4} = 4\left[\cancelto{1}{\tan\frac{\pi}{4}} - \cancelto{0}{\tan 0}\right]$$

$$= \boxed{4}$$

Some Properties of definite integrals:

Suppose $f(x)$ is a continuous function on $[a,b]$

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is an even function

4) $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function

even function $\Leftrightarrow f(-x) = f(x)$

odd function $\Leftrightarrow f(-x) = -f(x)$

Evaluate $\int_{-2}^2 (x^6 - x^4)dx$

$f(x) = x^6 - x^4$

$f(-x) = (-x)^6 - (-x)^4 = x^6 - x^4 = f(x) \Rightarrow$ even function

$\int_{-2}^2 (x^6 - x^4)\,dx = 2\int_0^2 (x^6 - x^4)dx = 2\left[\frac{x^7}{7} - \frac{x^5}{5}\right]\Big|_0^2$

$\vdots$

$= \boxed{}$

Evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\pi/4} \sin x \cdot x^4\,dx$

$f(x) = x^4 \sin x$

$f(-x) = (-x)^4 \sin(-x) = x^4 \cdot -\sin x = \underbrace{-x^4 \sin x}$

$= -f(x)$

$f(-x) = -f(x)$

odd function

$$\int_{-\pi/4}^{\pi/4} x^4 \sin x\,dx = \boxed{0}$$

Fundamental Theorem of Calculus

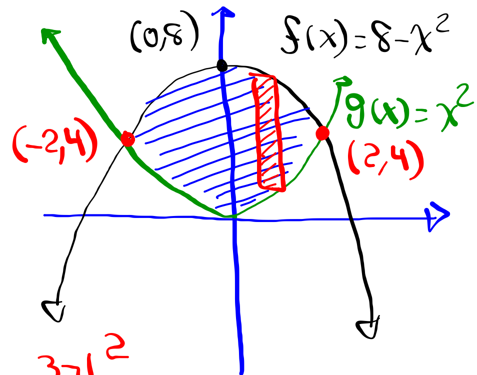
If f is cont. on $[a,b]$, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

ex: Find $\displaystyle\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$

Class QZ 14

Find the area enclosed by $f(x)=8-x^2$ and $g(x)=x^2$. Drawing Required.



$$A=\int_{-2}^{2}\left[8-x^2-x^2\right]dx$$

$$=2\int_0^2(8-2x^2)dx=2\left[8x-\frac{2x^3}{3}\right]\Bigg|_0^2$$

$$=2\left[8\cdot2-\frac{2\cdot2^3}{3}\right]=2\left[16-\frac{16}{3}\right]=2\cdot\frac{32}{3}=\boxed{\frac{64}{3}}$$